

1.1, 1.2 Completed Notes

1.1: Mathematics and Problem Solving

Four-Step Problem-Solving Process:

- Understand the problem.
 - What is the problem giving me?
 - What is the problem asking me?
- Devise a plan. Some strategies include:
 - Look for a pattern.
 - Make a table.
 - Draw a picture.
- Carry out the plan.
 - Follow your plan.
 - Check each step as you go.
- Look back.
 - Check your answer.
 - Did you answer the original question?
 - Was there another method?
 - Are there related problems for which you could use the same techniques?

Activity: Work with a partner to form a "book" in the following manner:

- Get 2 slips of paper from the table.
- Fold each piece of paper in half hamburger style, then put them together to look like a book.
- Close your book up, and number the outside page 1, flip and number page 2, page 3, etc.

When you are finished, answer the following questions.

- What is the total of the page numbers on one side of a sheet?

9 13

- What is the total of the page numbers on one whole sheet?

18 26

- What is the total of all the page numbers?

36 78

Repeat this for 3 slips of paper.

Answer the following questions.

- Predict the sum of the page numbers if we use 4 slips of paper.

One Side: 17 Total: 136

One Sheet: 34

- Without using a calculator, determine the value of $1 + 2 + \dots + 24$.

6 sheets

Total: 300

One side: 25

One sheet: 50

- Write a formula for the sum of the page numbers with n sheets.

One side: $4n+1$

One sheet: $2(4n+1)$

Total: $2 \cdot (4n+1) \cdot n = 2n(4n+1)$

Let's look at the actual problem solving approach:

Example: If you create a book out of n sheets, what is the sum of the page numbers?

Step 1 - Understand the Problem: What do we need to understand?

How to add. \Leftarrow define "sum". n ?

What is a book?

Step 2 - Devise a Plan: Our strategy was find a pattern.

How did we do it?

We tried 2, 3, 4, 6, etc.

Step 3 - Carry Out the Plan: Using the 2 sheet and 3 sheet example, we figured out how to determine the last page number and how to find (1) the total page numbers on each side of a sheet, (2) the total page numbers on each sheet, and (3) the sum of all page numbers. We then used this information to find the formula for the sum of the pages of a book made from n sheets.

Step 4 - Look Back:

We can check our answers for the 2 case: $1+2+3+4+5+6+7+8=36$.

Did we answer the question?

Yes

Let's try a different approach.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$$

$9 \times 4 = 36$

Example: What is the sum of the first n positive integers?

$$1 + 2 + 3 + \dots + (n-1) + n = S$$

$$n + (n-1) + (n-2) + \dots + 2 + 1 = S$$

$$(n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) = 2S$$

n

$$n(n+1) = 2S$$

$$S = \frac{n(n+1)}{2}$$

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Strategy: Examine a Related Problem

Example: What is the sum of the even numbers less than or equal to 40? Have we done a similar problem before?

$$2 + 4 + 6 + 8 + 10 + \dots + 38 + 40$$

42

$$42(10) = 420 \quad 10 \text{ pairs} \quad \frac{20(42)}{2}$$

$$2(1+2+3+4+\dots+20)$$

$$= 2\left(\frac{20(21)}{2}\right) = 420$$

Strategy: Make a Table

Example: A wealthy family hired a maid and a gardener. The maid comes in every 2 days, and the gardener comes in every 3 days. If they started on the same day, how many days will go by before they come in on the same day again?

	0	1	2	3	4	5	6	7	8
M	X		X		X		X		X
G	X			X			X		

They will meet every 6 days.

Strategy: Guess and Check

Example: Find two numbers whose product is 42 and whose sum is 17.

x	y
1	42
2	21
3	14
6	7

The two numbers are 3 and 14.

1. (Pigs and Chickens) A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, "I count 24 heads and 80 feet. How many pigs and how many chickens are there?"

P	C	
12	12	$12 \cdot 4 + 12 \cdot 2 = 72$
14	10	76
16	8	80

Strategy: Examine a Simpler Case

Example: If there are 10 people in the room and each person shakes every other person's hand, how many handshakes were performed? Nobody shakes hands twice, and shaking your own hand would make you look weird...

6 = 3 + 2 + 1

10 = 4 + 3 + 2 + 1

10 people: $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$

$$= \frac{9(9+1)}{2} = \frac{9(10)}{2} = 45$$

1. (Pigs and Chickens) A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, "I count 24 heads and 80 feet. How many pigs and how many chickens are there?"

$p + c = 24$

$4p + 2c = 80$

$p = 24 - c$

$4(24 - c) + 2c = 80$

$96 - 4c + 2c = 80$

$-2c = -16$

$c = 8$

$p = 16$

2. Arrange the numbers 1 through 9 into a square subdivided into nine smaller squares (like the one shown below), so that the sum of every row, column, and main diagonal is the same.

4	3	8
9	5	1
2	7	6

① $1 + 2 + 3 + \dots + 9 = \frac{9(9+1)}{2} = 45$

$\frac{45}{3} = 15$

② $1 + 5 + 9 = 15$ $2 + 4 + 9 = 15$ $3 + 4 + 8 = 15$

$1 + 6 + 8 = 15$ $2 + 5 + 8 = 15$ $3 + 5 + 7 = 15$

$2 + 6 + 7 = 15$ $4 + 5 + 6 = 15$

③ 5 is in 4 sums, so it's in the middle

2, 4, 6, 8 are in 3 sums, so these are the corners

put other numbers

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3. Take any number and add 15 to the number. Now multiply that by 4. Next subtract 8 and divide the difference by 4. Now subtract 2 from the quotient and write down the answer. Your professor can tell you the original number. How is your professor find the original number so quickly?

$$\begin{array}{lcl}
 1+15=16 & 5+15=20 & \text{Result is one} \\
 16 \times 4 = 64 & 20 \times 4 = 80 & \text{higher than} \\
 64 - 8 = 56 & 80 - 8 = 72 & \text{original} \\
 56 \div 4 = 14 & 72 \div 4 = 18 & \\
 14 - 12 = 2 & 18 - 12 = 6 &
 \end{array}$$

$$\begin{aligned}
 & \frac{4(x+15)-8}{4} - 12 = \frac{4x+60-8}{4} - 12 \\
 & = \frac{4x+52}{4} - 12 = x+13-12 = x+1
 \end{aligned}$$

4b. If the number of rows and columns of the checkerboard is doubled (so you have a 16 by 16 board), is the number of squares doubled? Justify your answer.

There are $16 \times 16 = 256$ 1 by 1 squares, which is not 64×2 .

In general, we also have overlaps and larger squares.

$$1^2 + 2^2 + 3^2 + \dots + 16^2 = \frac{16(17)(33)}{6} = 1496$$

Strategy: Work Backward

Example: You have an 80 average on 6 quizzes. Your teacher tells you that you can drop your lowest quiz grade of 30. What is your new average?

$$\begin{aligned}
 80 &= \frac{Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + 30}{6} & \text{ave: } \frac{450}{5} \\
 480 &= Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + 30 & = 90 \\
 450 &= Q_1 + Q_2 + Q_3 + Q_4 + Q_5
 \end{aligned}$$

Strategy: Use Direct Reasoning.

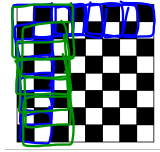
Example: If two people won 3 games of checkers each, what is the minimum number of games played?

AABABB 6 games

Strategy: Write an Equation. This was discussed heavily in Math 111.

4. How many squares can be found in the 8 by 8 checkerboard figure below?

$$\begin{aligned}
 1 \text{ by } 1: & 8 \times 8 = 64 \\
 2 \text{ by } 2: & 7 \times 7 = 49 \\
 3 \text{ by } 3: & 6 \times 6 = 36 \\
 4 \text{ by } 4: & 5 \times 5 = 25 \\
 5 \text{ by } 5: & 4 \times 4 = 16 \\
 6 \text{ by } 6: & 3 \times 3 = 9 \\
 7 \text{ by } 7: & 2 \times 2 = 4 \\
 8 \text{ by } 8: & 1 \times 1 = 1
 \end{aligned}$$

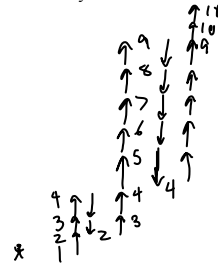


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Strategy: Identify a sub-goal. In some problems, you may want to find a piece of information that will help you solve the problem first.

Strategy: Make a diagram.

Example: To get some cardio exercise, you climb the stairs in a tall building. You start from the first floor (Floor 1). You then go up 3 floors, down 2 floors, up 7 floors, down 5 floors, and then up 7 floors to stop at the top floor. How many floors does the building have?



11 floors

$$1 + 3 - 2 + 7 - 5 + 7$$

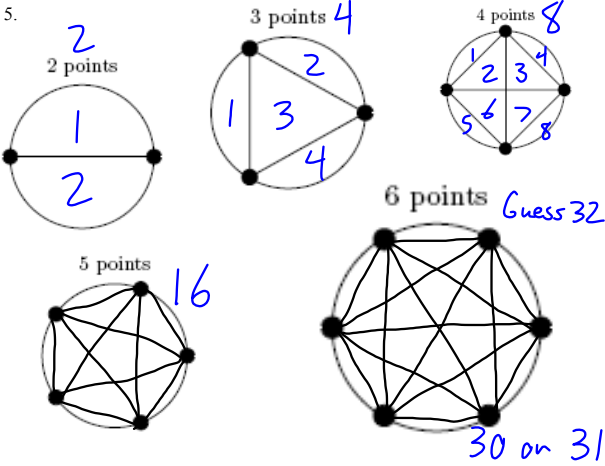
Strategy: Use Indirect Reasoning. Sometimes it is easier to show what the opposite can't happen. An important form of this is the process of elimination.

Example: Andrew, Michael, and Travis played a strategy game. Michael did not come in first place, as usual. Travis beat Michael but he did not come in first place. Who took first, second, and third place?

	1st	2nd	3rd
A	O	X	X
M	X	X	O
T	X	O	X

1st place - Andrew
2nd - Travis
3rd - Michael

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1.2: Patterns

What appears to be a pattern may need further checking. You need enough data to identify a pattern.

Example: Fill in the following pattern: 13, 19, 23, 31, 37, 43

Example: Fill in 2, 4, 8, 16, 32 in as many ways as you can think of.

2, 4, 6, 8, 10

2, 4, 16, 25, 6, ...

2, 4, 2, 4, 2, 4, ...

2, 4, 7, 11, 16, ...

2, 4, 6, 10, 16, ...